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ORIGINAL PAPER

A coupled Ramani equation: multiple soliton solutions

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Abstract In this work, a sixth-order coupled Ramani equation is investigated. The simplified form of the Hirota's method is used for analytic treatment of this equation. The constraint condition between coefficients of the spatial and the temporal variables is treated. Multiple soliton solutions and multiple singular soliton solutions are formally derived for this model.

Keywords Coupled Ramani equation · Simplified Hirota's method · Multiple soliton solutions

1 Introduction

In [1–8], the sixth-order nonlinear Ramani equation, that reads

$$\begin{aligned} &u_{xxxxxx} + 15u_x u_{xxxx} + 15u_{xx}u_{xxx} + 45u_x^2 u_{xx} - 5(u_{xxxt} \\ &\quad +3u_x u_{xt} + 3u_t u_{xx}) - 5u_{tt} = 0. \end{aligned} \tag{1}$$

was investigated for integrability, Lax pair, Bäcklund transformation. In [2], the method of truncated singular expansion was used to derive Lax pair and Bäcklund self-transformation for Eq. (1). In [4], Lax pairs and Bäcklund transformations were used to handle (1).

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However, a specific case of a coupled Ramani equation given by

$$u_{xxxxxx} + 15u_x u_{xxxx} + 15u_{xx}u_{xxx} + 45u_x^2 u_{xx} - 5(u_{xxxt} + 3u_x u_{xt} + 3u_t u_{xx}) - 5u_{tt} + 18w_x = 0, \tag{2}$$

$$w_t - w_{xxx} - 3w_x u_x - 3wu_{xx} = 0. \tag{3}$$

was investigated in [4–8] using different approaches.

Recently, a coupled Ramani equation of the form [1–8]

$$u_{xxxxxx} + 15u_{xx}u_{xxxx} + 15u_{xx}^3 - 5u_{xxxt} - 15u_{xt}u_{xx} - 5u_{tt} + 18(p_x - 3w_x v + 3wv_x) = 0, \tag{4}$$

$$p_t - p_{xxx} - 3u_{xx}p_x + 3w_{xx}v_x - 3w_x v_{xx} = 0, \tag{5}$$

$$v_t - v_{xxx} - 3u_{xx}v_x = 0, \tag{6}$$

$$w_t - w_{xxx} - 3u_{xx}w_x = 0, \tag{7}$$

was investigated by using Bäcklund transformation and Lax pair of this coupled equation.

In a new coupled Ramani equation was proposed in the form [1–8]

$$\begin{aligned} &u_{tt} - u_{xxxxxt} - \frac{25}{3}u_{xt}u_{xxx} - 5u_x u_{xxxt} - 5u_x^2 u_{xt} - \frac{10}{3}u_{xxxx}u_t - 10u_{xx}u_{xxt} \\ &-10u_x u_{xx}u_t - \frac{10}{3}vu_{xt} - \frac{10}{3}v_x u_t = 0, \\ &v_t - \frac{1}{2}(u_{xxxt} + 3u_{xx}u_t + 3u_x u_{xt}) = 0. \end{aligned} \tag{8}$$

Bilinear equations for Eq. (8) were given by using a dependent variable transformation for u and v. Moreover, the Lax pair of (8) were given by

$$\begin{aligned} &\psi_{xxxxxx} + 5u_x\psi_{xxxx} + 10u_{xx}\psi_{xxx} + \left(\frac{25}{3}u_{xxx} + 5u_x^2 + \frac{10}{3}v\right)\psi_{xx} \\ &+\left(\frac{10}{3}u_{xxxx} + 10u_x u_{xx} + \frac{10}{3}v_x\right)\psi_x + u_t\psi - \mu\psi_x = 0, \\ &\psi_t - \psi_{xxxxx} - 5u_x\psi_{xxx} - 5u_{xx}\psi_{xx} - \left(\frac{10}{3}u_{xxx} + 5u_x^2 + \frac{10}{3}v\right)\psi_x = 0. \end{aligned} \tag{9}$$

Explicit soliton solutions were not derived in [4], instead it was reported that such solutions can be obtained. However, in [5], the method of dynamical systems was used to derive travelling wave solutions.

We aim in this work to formally derive the multi-kink solutions and the multi-soliton solutions for u and v of the modified coupled Ramani equation (8) to show its complete integrability that was confirmed in [4]. The simplified form of Hirota's method [9–21] will be used to achieve this goal. The simplified Hirots's method does not depend on the construction of the bilinear forms, instead it assumes the multi-soliton solutions can be expressed as polynomials of exponential functions [11]. The multiple regular soliton solutions and multiple singular soliton solutions for (8) will be derived using this simplified method.

Conducting polymers, in the polymer chemistry, have received considerable research work because of their unique features. It is to be noted that solitons appear in many chemical applications such as electrically conducting polyenes, Polarons, bipolarons,and bolitons in conducting polymers as examined in [22,23]. In [23], the solitons, polarons and excitations in polyacetylene were examined. Polarons and bipolarons are studied as soliton–antisoliton pairs. At low doping levels, charged solitons are formed either directly from existing neutral solitons or as the result of recombination among polarons [22,23]. The solitons in radiation chemistry was thoroughly examined in [24,25]. In summary, it is noted that many of the solitons, both topological and non-topological can be used to describe vector polarons. This confirms the fact that solitons, and multi-solitons are heavily used in chemical applications.

2 A modified coupled Ramani equation

In this section we will study a new form of a coupled Ramani equation given by

$$\begin{aligned} &u_{tt} - u_{xxxxxt} - \frac{25}{3}u_{xt}u_{xxx} - 5u_x u_{xxxt} - 5u_x^2 u_{xt} - \frac{10}{3}u_{xxxx}u_t - 10u_{xx}u_{xxt} \\ &\quad -10u_x u_{xx}u_t - \frac{10}{3}vu_{xt} - \frac{10}{3}v_x u_t = 0, \\ &\quad v_t - \frac{1}{2}\left(u_{xxxt} + 3u_{xx}u_t + 3u_x u_{xt}\right) = 0. \end{aligned} \tag{10}$$

2.1 Multiple-soliton solutions

We first introduce an auxiliary variable z and set the new dependent variables

$$\begin{aligned} u(x,z,t) &= R\ln(f(x,z,t))_x, \\ v(x,z,t) &= \left(\frac{f_z}{f}\right)_x. \end{aligned} \tag{11}$$

Using the assumption

$$u(x,z,t) = e^{k_i x + r_i z - \omega_i t}, \tag{12}$$

into the linear terms of (10) gives the dispersion relation by

$$\omega_i = -k_i^5. \tag{13}$$

This in turn gives the following phase variable

$$\theta_i = k_i x + r_i z + k_i^5 t, \quad i = 1, 2, 3. \tag{14}$$

Combining (11)–(14) and substituting the outcome in (10) results in a constraint condition that solitons are guaranteed only if we set

$$r_i = k_i^3, \quad i = 1, 2, 3. \tag{15}$$

The multi soliton solutions of (10) are assumed above in (11), where the auxiliary function $f(x, z, t)$, for the single soliton solution, is given by

$$f(x, z, t) = 1 + e^{\theta_1} = 1 + e^{k_1 x + k_1^3 z + k_1^5 t}. \tag{16}$$

Substituting (16) into (10) and solving for R we find

$$R = 2. \tag{17}$$

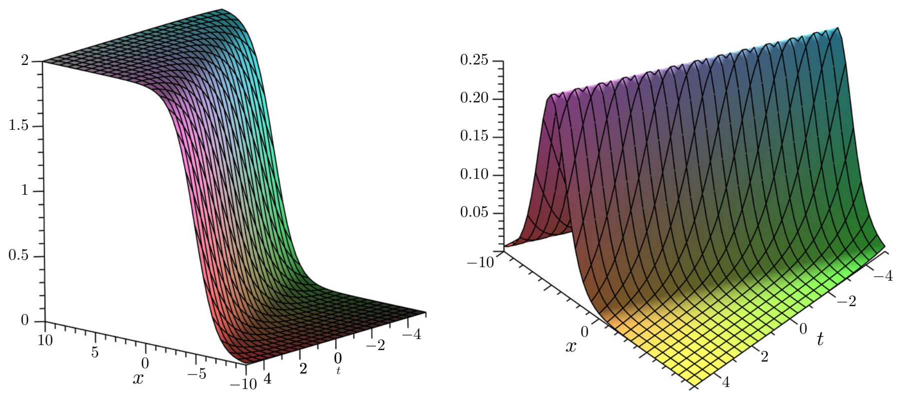
This in turn the single kink and the single soliton solutions

$$\begin{aligned} u(x, z, t) &= \frac{2k_1 e^{k_1 x + k_1^3 z + k_1^5 t}}{1 + e^{k_1 x + k_1^3 z + k_1^5 t}}, \\ v(x, z, t) &= \frac{k_1^4 e^{k_1 x + k_1^3 z + k_1^5 t}}{(1 + e^{k_1 x + k_1^3 z + k_1^5 t})^2}, \end{aligned} \tag{18}$$

respectively, are readily obtained. Figure 1 below shows the single kink solution for $u(x, z, t)$ and the one soliton solution for $v(x, z, t)$.

For the two soliton solutions we set the auxiliary function by

$$f(x, z, t) = 1 + e^{k_1 x + k_1^3 z + k_1^5 t} + e^{k_2 x + k_2^3 z + k_2^5 t} + a_{12} e^{(k_1+k_2)x + (k_1^3+k_2^3)z + (k_1^5+k_2^5)t}. \tag{19}$$



(a) **(b)**

Fig. 1 **a** One-kink solution with $-5 \le t \le 5, -10 \le x \le 10$, **b** one-soliton solution with $-5 \le t \le 5, -10 \le x \le 10$

Substituting (19) in (10), and proceeding as before we obtain the phase shift a_{12} by

$$a_{12}=\frac{k_1^6-k_1^5k_2-k_1k_2^5+k_2^6}{k_1^6+k_1^5k_2+k_1k_2^5+k_2^6}. \tag{20}$$

and hence we set

$$a_{ij}=\frac{k_i^6-k_i^5k_j-k_ik_j^5+k_j^6}{k_i^6+k_i^5k_j+k_ik_j^5+k_j^6},\quad 1\leq i<j\leq 3. \tag{21}$$

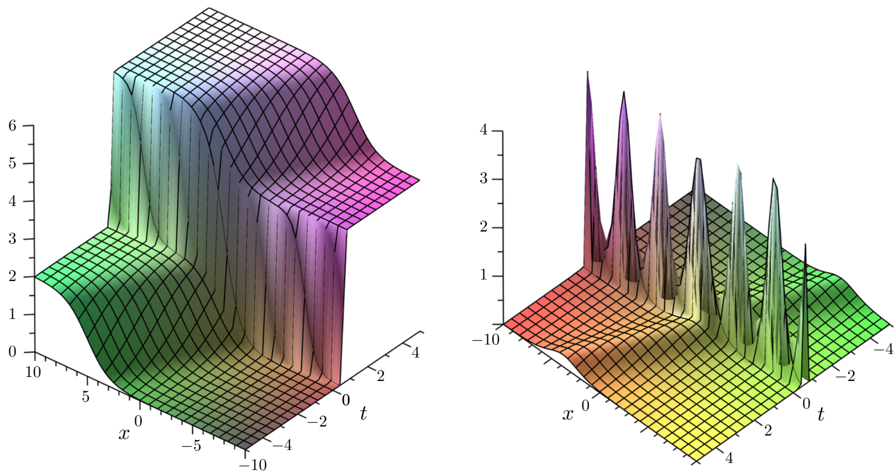
The two kink solutions for $u(x,z,t)$ and the two soliton solutions for $v(x,z,t)$ are obtained by substituting (20) and (19) into (11).

It is interesting to point out that the coupled Ramani equation (10) does not show any resonant phenomenon because the phase shift term a_{12} in (20) cannot be 0 or ∞ for $|k_1|\neq|k_2|$.

Figure 2 below shows the two-kink solution for $u(x,z,t)$ and the two-soliton solution for $v(x,z,t)$.

For the three soliton solutions, we set the auxiliary function by

$$\begin{aligned} f(x,z,t)=1&+e^{k_1x+k_1^3z+k_1^5t}+e^{k_2x+k_2^3z+k_2^5t}+e^{k_3x+k_3^3z+k_3^5t}\\ &+a_{12}e^{(k_1+k_2)x+(k_1^3+k_2^3)z+(k_1^5+k_2^5)t}\\ &+a_{13}e^{(k_1+k_3)x+(k_1^3+k_3^3)z+(k_1^5+k_3^5)t}+a_{23}e^{(k_2+k_3)x+(k_2^3+k_3^3)z+(k_2^5+k_3^5)t}\\ &+b_{123}e^{(k_1++k_2k_3)x+(k_1^3+k_2^3+k_3^3)z+(k_1^5+k_2^5+k_3^5)t}. \end{aligned} \tag{22}$$



(a) **(b)**

Fig. 2 **a** Two-kink solution with $-5\leq t\leq 5$, $-10\leq x\leq 10$, **b** two-soliton solution with $-5\leq t\leq 5$, $-10\leq x\leq 10$

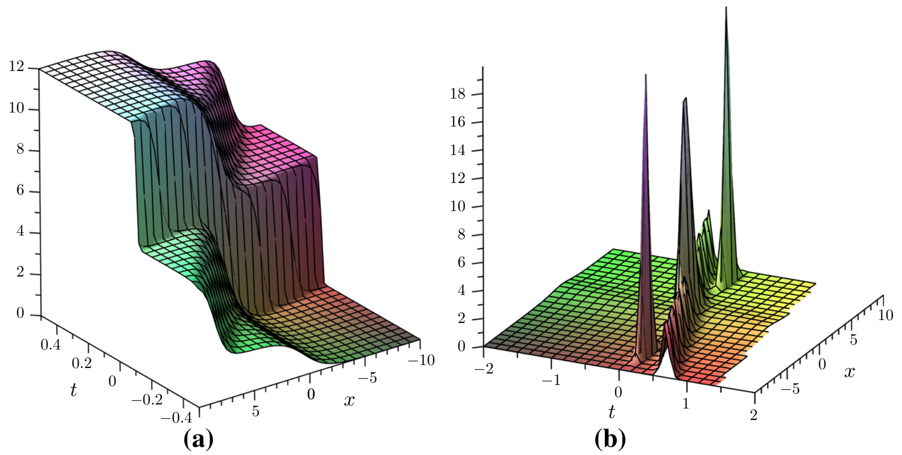


Fig. 3 **a** Three-kink solution with $-0.5 \leq t \leq 0.5, -10 \leq x \leq 10$, **b** three-soliton solution with $-2 \leq t \leq 2, -10 \leq x \leq 10$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{23}$$

The three kink solutions for $u(x, z, t)$ and the three soliton solutions for $v(x, z, t)$ are obtained by substituting (22) into (10). It was stated before, the integrability of the coupled Ramani equation (10) was formally proved in [4]. The results obtained above confirm that the new coupled Ramani equation (10) is completely integrable and N-soliton solutions can be obtained for finite N, where $N \geq 1$,

Figure 3 below shows the three-kink solution for $u(x, z, t)$ and the three-soliton solution for $v(x, z, t)$.

2.2 Multiple singular-soliton solutions

To determine one-soliton, two soliton and three-soliton solutions we set the auxiliary functions in the forms

$$f(x, z, t) = 1 - e^{k_1 x + k_1^3 z + k_1^5 t} \tag{24}$$

$$f(x, z, t) = 1 - e^{k_1 x + k_1^3 z + k_1^5 t} - e^{k_2 x + k_2^3 z + k_2^5 t} + a_{12} e^{(k_1+k_2)x + (k_1^3+k_2^3)z + (k_1^5+k_2^5)t}, \tag{25}$$

and

$$\begin{aligned} f(x, z, t) = 1 &- e^{k_1 x + k_1^3 z + k_1^5 t} - e^{k_2 x + k_2^3 z + k_2^5 t} - e^{k_3 x + k_3^3 z + k_3^5 t} \\ &+ a_{12} e^{(k_1+k_2)x + (k_1^3+k_2^3)z + (k_1^5+k_2^5)t} \\ &+ a_{13} e^{(k_1+k_3)x + (k_1^3+k_3^3)z + (k_1^5+k_3^5)t} + a_{23} e^{(k_2+k_3)x + (k_2^3+k_3^3)z + (k_2^5+k_3^5)t} \end{aligned}$$

$$-a_{12}a_{13}a_{23}e^{(k_1++k_2k_3)x+(k_1^3+k_2^3+k_3^3)z+(k_1^5+k_2^5+k_3^5)t}. \tag{26}$$

respectively. We follow the same analysis presented above, hence we skip details. Using (11), the following one-singular kink and the one-soliton solutions are given by

$$\begin{aligned} u(x,z,t) &= -\frac{2k_1e^{k_1x+k_1^3z+k_1^5t}}{1-e^{k_1x+k_1^3z+k_1^5t}},\\ v(x,z,t) &= -\frac{k_1^4e^{k_1x+k_1^3z+k_1^5t}}{(1-e^{k_1x+k_1^3z+k_1^5t})^2}, \end{aligned} \tag{27}$$

respectively. Multiple singular solutions can be obtained in a like manner.

3 Discussion

A new integrable coupled Ramani equation was investigated. The simplified form of the Hirota's bilinear method was applied to conduct the analysis set for this work. Multiple soliton solutions and multiple singular soliton solutions were obtained. The resonance relation does not exist for this coupled Ramani equation.

It is to be noted that solitons are used in conducting polymers in the polymer chemistry. Solitons appear in many chemical applications such as electrically conducting polyenes, Polarons, bipolarons,and bolitons in conducting polymers as examined in [22,23]. This confirms the fact that solitons, and multi-solitons are heavily used in chemical applications.

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